Reynolds Shear Stress and Heat Flux Balance in a Turbulent Round Jet

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Measurements are presented of the budgets of the turbulent energy, Reynolds shear stress, and the heat flux in an axisymmetric heated turbulent jet with a coflowing isothermal external stream. The jet-to-external-velocity ratio is 6.6, and the temperature of the jet is 34° C above that of the ambient temperature external stream. The budgets are obtained for both the conventional and turbulent zone averages of the various quantities. The conventional budgets of the Reynolds shear stress and the axial and radial heat flux show that the production of these quantities is balanced effectively by the terms containing the pressure fluctuations. The turbulent zone budget of the shear stress and radial heat flux show enhanced diffusion of these quantities by the radial velocity fluctuations in the outer part of the jet. The measured values that feature in the budgets together with the distributions, obtained by difference, of the pressure-containing terms are used to test some of the assumptions made by Donaldson for calculating turbulent free shear layers.

Nomenclature

a_1	= structure function $\overline{uv}/\overline{q^2}$
$a_{1\theta}$	= structure function defined by Eq. (15)
G	= diffusion function $\frac{1}{2}\overline{q^2v}/\overline{uv}^{3/2}_{max}$
G_θ	= diffusion function defined by Eq. (16)
I	= intermittency function, equal to one (turbulent flow) or zero (irrotational flow)
L_x	= integral length scale of fluctuating quantity x
L_0	= length scale defined as the value of r at which $U-U_1=U_0/2$
L_ϵ	= energy dissipation length scale $\overline{uv}^{3/2}/\epsilon$
L_χ	= temperature dissipation length scale $\overline{v\theta^2}/(\overline{uv}^{1/2}\chi)$
ℓ	= mixing length $\overline{uv}^{1/2}/(\partial U/\partial r)$
ℓ_K	= Kolmogorov microscale $(\nu^3/\epsilon)^{1/4}$
ℓ_θ	= thermal mixing length $\overline{v\theta}/(\overline{uv}^{1/2}\partial T/\partial r)$
p	= kinematic pressure fluctuations
q^2	= turbulent energy $(=u^2+v^2+w^2)$
r	= radial coordinate
R_λ	= turbulence Reynolds number $\overline{u^2}^{1/2}\lambda/\nu$
T	= difference between local mean temperature and T_1
T_0	= value of T on the axis of symmetry
T_1	= ambient temperature of external stream
U, V	= mean velocities in axial and radial directions, respectively
u,v,w	= velocity fluctuations in axial, radial, and azimuthal directions, respectively
$\overline{uv}$	= Reynolds shear stress
$\overline{u\theta}$	= axial heat flux
$\overline{v\theta}$	= radial heat flux
U_0	= difference between velocity U at the axis of symmetry and U_1
x	= distance downstream from nozzle exit
α	= thermal diffusivity
ϵ	= dissipation of turbulent energy
η	= nondimensional coordinate $(=r/L_0)$
θ	= temperature fluctuation
Λ_1	= length scale defined by Eq. (8)
Λ_2	= diffusion length scale defined by Eqs. (9) or (10)
Λ_3	= diffusion length scale defined by Eq. (11)
λ	= Taylor microscale
ν	= kinematic viscosity
ρ	= density
ϕ	= azimuthal direction
χ	= dissipation of temperature fluctuations
$\overline{(\ )}$	= conventional time average of ( )
$(\ )_t$	= turbulent zone average of ( )

Subscripts

i,j,k	= Cartesian tensor indices
t	= refers to turbulent quantity only
u	= relates to velocity u
θ	= relates to temperature θ

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Index categories: Jets, Wakes, and Viscid-Inviscid Flow Interactions; Boundary Layers and Convective Heat Transfer—Turbulent.

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‡In the calculation of the advection term in Eqs. (1-4), V and the term involving a gradient in the x direction were obtained by assuming self-preserving distributions of the mean velocity U and of the turbulent energy, Reynolds shear stress, and heat fluxes.

I. Introduction

In a previous paper,[1] we presented measurements of the properties of both mean and fluctuating velocity and temperature fields in a slightly heated axisymmetric turbulent jet with a coflowing isothermal external stream. The conditional sampling technique using the temperature fluctuation θ as a detector of the turbulent/irrotational interface was adopted to measure quantities such as the Reynolds stresses and heat fluxes in the turbulent part only of the flow. In particular, the budget of $\overline{\theta^2_t}$, the average of the squared temperature fluctuations in the turbulent flow, was obtained by measuring the production, advection, diffusion, and dissipation of $\overline{\theta^2_t}$. Unlike the budget of the turbulent energy, the $\overline{\theta^2}$ or $\overline{\theta^2_t}$ budgets do not contain the pressure fluctuation term. As the measured $\overline{\theta^2_t}$ budget was in fairly close balance, it was felt that measurements of the budgets of the heat fluxes $\overline{\theta u}$ (u is the axial velocity fluctuation) and $\overline{\theta v}$ (v is the radial velocity fluctuation) could shed some light on the poorly understood pressure temperature gradient correlations that do occur in the equations for $\overline{\theta u}$ and $\overline{\theta v}$ (see, for example, Ref.2).

In this paper the budgets for the conventional and turbulent zone averages of the instantaneous heat fluxes θu, θv, and Reynolds shear stress uv are presented in Secs. III and IV. Some of the terms in these budgets then are used in Sec. V to test the major assumptions in the method developed by Donaldson[3] to calculate turbulent shear flows. A few parameters related to the structure of the turbulence in a heated jet and first used for a nonisothermal boundary layer[4] also are presented in Sec. V and compared with the corresponding parameters in the axisymmetric wake of a heated sphere.

II. Experimental Techniques and Conditions

A detailed description of the experimental jet facility and techniques has been given in Refs. 1 and 5 and will not be repeated here. The fluctuations u, v, θ were obtained with a three-wire arrangement. A DISA X wire (5-μm-diam Pt-coated tungsten wires) was used for the measurement of u and v and a 1-μm platinum wire, separated 1 mm from the X wire, was used to measure θ. The contamination of the X wire signals by the temperature fluctuation was removed with the use of components on the EAI 180 analog computer by the method described in Ref. 1. The temperature signal was used to generate the intermittency function $I(t)$, which was set to one when the instantaneous temperature exceeded the external air ambient temperature by an amount set by a threshold level and set to zero when the threshold was not exceeded. The choice of threshold level and the sensitivity of the conditional averages obtained with the use of $I(t)$ were discussed in Ref. 1. Turbulent zone averages of quantities involving u, v, θ were obtained with a P.A.R. boxcar integrator (model CW-1) by averaging the quantity of interest only when $I(t)$ was equal to one. Turbulent zone averages presented in the following sections are identified by a subscript t and are such that $\overline{q_t}=\overline{Iq}/\overline{I}$, where q stands for either u, v, θ, or a combination of these quantities. The averages $\overline{q_t}$ were obtained for records of approximately 100-sec duration.

The jet nozzle diameter was 2.03 cm, and the temperature of the jet was approximately 34° above that of the external co-flowing stream. The jet velocity was equal to 32 msec^{-1}, and the jet-to-external-velocity ratio was maintained at 6.6. All of the measurements presented in the following section were obtained at a station 59 diam downstream of the nozzle exit, where the velocity defect U_0 (difference between the velocity on the axis and the external velocity) was 3.1 msec^{-1} and the corresponding temperature defect T_0 was 2.43°C. This value of T_0 is sufficiently small for the temperature to be considered as a passive contaminant of the flow. The half temperature radius L_0, defined as the distance from the axis to the position where the local temperature defect is equal to $T_0/2$, is approximately 6.4 cm. The Reynolds number U_0L_0/ν is equal to 1.35×10^4 at the measuring station.

III. Turbulent Energy and Reynolds Shear Stress Budgets

The equation for the turbulent energy $\overline{q^2}$ $(=u^2+v^2+w^2)$ can be written approximately as

$$U\frac{\overline{\partial q^2}}{\partial x}+V\frac{\overline{\partial q^2}}{\partial r}=\overline{uv}\frac{\partial U}{\partial r}+[1/r]\frac{\partial}{\partial r}[r(\overline{q^2v}+\overline{pv})]+\epsilon \tag{1}$$

where ϵ is the dissipation of the turbulent energy, approximated here by the isotropic relation $\epsilon=15\nu\overline{(\partial u/\partial x)^2}$. As w was not measured, $\overline{q^2}$ has been approximated by $3(\overline{u^2}+\overline{v^2})/2$. The mean radial velocity V was calculated from the continuity equation

$$V/r+\partial U/\partial x+\partial V/\partial r=0$$

The various terms in Eq. (1) have been normalized with the velocity scale U_0 and length scale L_0 and are shown in Fig. 1a. The advection‡ is approximately equal to the dissipation in the region $\eta<1$. The production term tends to balance the dissipation for $\eta>1$. The peak in the production represents a significant proportion ($\sim70\%$) of the maximum advection. This is not too different from the result obtained by Wyganski and Fiedler[6] for the self-preserving axisymmetric jet in still air, but it is in clear contrast with the energy budget of Uberoi and Freymuth[7] in the axisymmetric wake of a sphere where the maximum advection is about 10 times the maximum production. A major difference between the present results and those in the self-preserving jet[6] is the distribution of the diffusion terms. Whereas the diffusion by $\overline{q^2v}$ shows a loss near the axis in the self-preserving jet, the present distribution indicates a substantial gain. Also, the present diffusion by $\overline{pv}$ is of opposite sign to that by $\overline{q^2v}$ at least near the axis and the outer edge, whereas in the self preserving jet, the diffusion by $\overline{pv}$ is essentially of the same sign as that by $\overline{q^2v}$ throughout the section. It should be noted of course, that the diffusion by $\overline{pv}$ was obtained by difference, both in the present results and in the self-preserving jet and therefore reflects inaccuracies in the measurements of all of the other quantities, in particular the third-order moments of $\overline{q^2v}$ and the assumed isotropy of ϵ. It is also interesting to note that the present distribution of diffusion by $\overline{q^2v}$ is qualitatively similar to that of diffusion by $\overline{\theta^2v}$ for the budget of $\overline{\theta^2}$ in Ref.1.

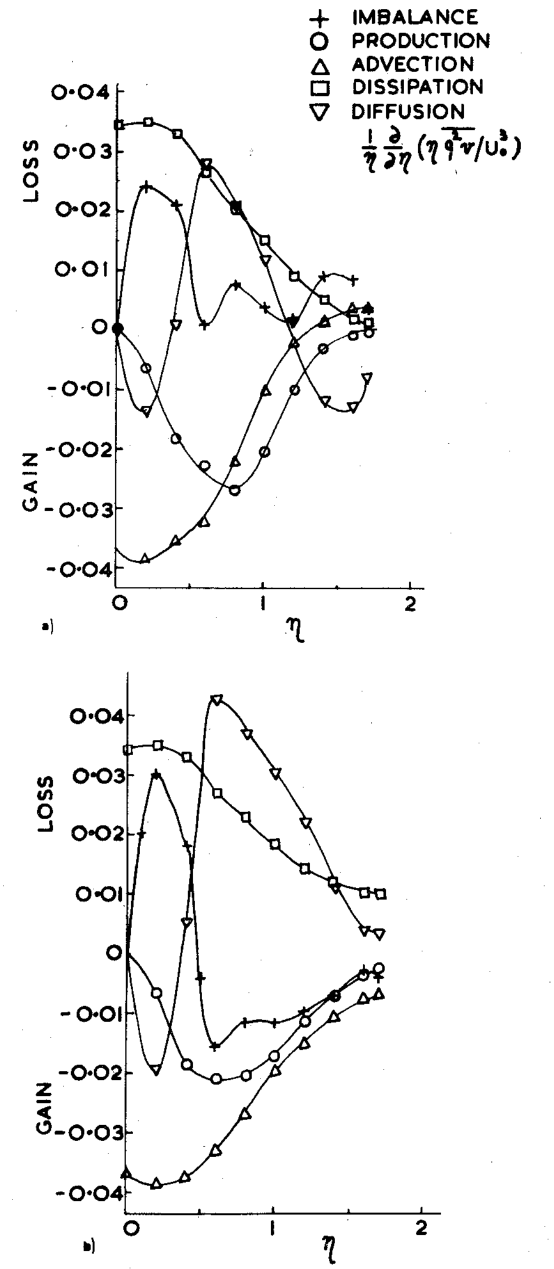

Fig. 1 Budgets a) turbulent energy $\overline{q^2}$ and b) $\overline{q_t^2}$. (Note that imbalance includes diffusion by $\overline{pv}$ by difference.)

The various terms of Eq. (1) measured in the turbulent region only of the flow are shown, again in normalized form, in Fig. 1b. The main difference between these results and the conventional budge of Fig. 1a is the shape in the diffusion curves. The diffusion by $(\overline{q^2 v})_t$ continues to show a loss toward the edge of the jet and is now essentially of opposite sign to the diffusion by $(\overline{pv})_t$. The other point of interest is that the dissipation remains sufficiently large near the edge of the jet to balance the gain of energy by production and advection, but the contribution from the diffusion is negligible.

The equation for the Reynolds shear stress $\overline{uv}$ can be written approximately as (in the cylindrical coordinate system)

$$U\frac{\partial\overline{uv}}{\partial x}+V\frac{\partial\overline{uv}}{\partial r}+\overline{v^2}\frac{\partial U}{\partial r}+(1/r)\frac{\partial}{\partial r}(r\overline{uv^2})-\frac{\overline{w^2u}}{r}$$

$$=-\left[\overline{u\frac{\partial p}{\partial r}+v\frac{\partial p}{\partial x}}\right]+\nu(\overline{u\nabla^2 v}+\overline{v\nabla^2 u}-\overline{uv}/r^2$$

$$-(2/r^2)\overline{u\partial w/\partial\phi} \qquad (2)$$

Most of the terms in Eq. (2) are shown in Fig. 2a, nondimensionalized by U_0 and L_0. The advection of $\overline{uv}$ [first two terms on the left-hand side of (2)] is of opposite sign to the diffusion (last two terms on the left-hand side). The viscous term can be reduced, using isotropy and the boundary-layer approximations, to $\nu\,\overline{(\partial u/\partial x)\,(\partial v/\partial x)}$, which is required to be zero by isotropy. The velocity pressure gradient correlation terms, obtained here by difference, thus are required to balance the production term [third term on the left-hand side of Eq. (2)] and act as a sink for the production of $\overline{uv}$. A similar result has been obtained by Wyngaard et al.[8] for the approximately two-dimensional atmospheric boundary layer under a wide range of stability conditions. There is also evidence for the importance of the pressure terms in the budget for the normal Reynolds stress $\overline{u^2}$ from the directly measured pressure fluctuation obtained by Elliott[9] in the marine boundary layer under neutral conditions. The budget of $\overline{uv}$ for only the turbulent part of the flow (Fig. 2b) shows a marked flux of $\overline{uv}$ toward the outer part of the jet, resulting in a local peak near $\eta=1.6$ of the pressure gradient-velocity interaction term.

IV. Budgets of Heat Fluxes $\overline{\theta u}$ and $\overline{\theta v}$

The equations for the heat flux vector have been written by Corrsin[2] in the Cartesian coordinate system. In the cylindrical coordinate system, the equation for the radial heat flux $\overline{\theta v}$ can be approximated, to the boundary-layer approximation, by

$$U\frac{\partial\overline{v\theta}}{\partial x}+V\frac{\partial\overline{v\theta}}{\partial r}+\overline{v^2}\frac{\partial T}{\partial r}+[1/r]\frac{\partial}{\partial r}(r\overline{v^2\theta})-\frac{\overline{w^2\theta}}{r}$$

$$=-\overline{\theta\frac{\partial p}{\partial r}}+\nu\left[\frac{\partial^2\overline{v\theta}}{\partial r^2}+[1/r]\frac{\partial\overline{v\theta}}{\partial r}-\frac{\overline{\theta v}}{r^2}\right.$$

$$\left.-6\overline{\frac{\partial v}{\partial x}\frac{\partial\theta}{\partial x}}-[2/r^2\theta]\overline{\frac{\partial w}{\partial\phi}}\right] \qquad (3)$$

The terms in this equation are normalized by U_0, L_0, and the temperature defect T_0 and shown in Fig. 3. The diffusion terms [fourth and fifth terms on the left-hand side of Eq. (3)] can be approximated roughly by $\partial(\overline{v^2\theta})/\partial r$, the transport of the flux $\overline{\theta v}$ by the radial velocity fluctuations, provided that $\overline{v^2\theta}\simeq\overline{w^2\theta}$. To simplify the viscous term, it was assumed here that the thermal diffusivity α and moment diffusivity υ were equal. Apart from the last term in the brackets on the right-hand side of Eq. (3), most of the contributions from the viscous term were measured and found to be small. The viscous term $6\nu\overline{(\partial v/\partial x)\,(\partial\theta/\partial x)}$, which is required to be zero by isotropy, was obtained from the correlation $\overline{(\partial v/\partial t)}$ $\overline{(\partial\theta/\partial t)}$ using Taylor's hypothesis and, as seen in Fig. 3a, is negligible. The distributions of the terms in Fig. 3a bears a strong analogy to those in the $\overline{uv}$ budget. Again, the temperature-pressure gradient correlation is required to cancel the production of $\overline{v\theta}$. The budget of Eq. (3) in the turbulent flow region (Fig. 3b) reveals the important and almost constant contribution from the diffusion term in the region $\eta<1.4$. The almost constant transport of $(\overline{v\theta})_t$ by the turbulent v fluctuations follows from the linear distribution of $(\overline{v^2\theta})_t$ in Fig. 6. For $\eta>1.6$, there is a significant gain by the diffusion term, similar to that in the $(\overline{uv})_t$ budget, and a corresponding increase in the $(\overline{\theta\partial p/\partial r})_t$ term.

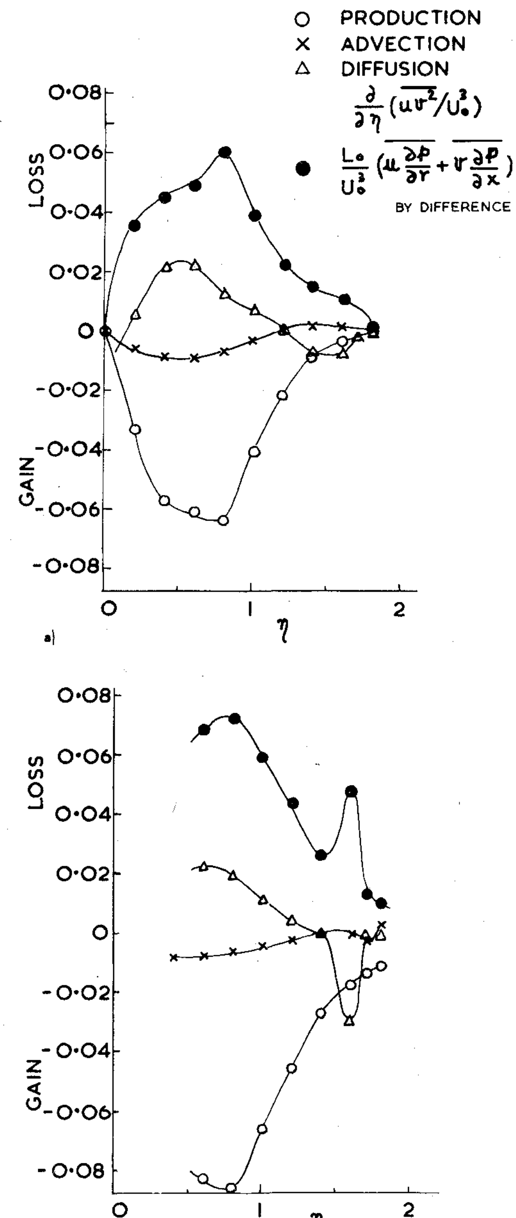

Fig. 2 Budgets: a) Reynolds stress $\overline{uv}$ and b) $\overline{uv}_t$.

The equation for $\overline{u\theta}$ can be approximated by

$$U\frac{\partial\overline{\theta u}}{\partial x}+V\frac{\partial\overline{\theta u}}{\partial r}+\overline{v\theta}\frac{\partial U}{\partial r}+\overline{uv}\frac{\partial T}{\partial r}+[1/r]\frac{\partial}{\partial r}(r\overline{uv\theta})$$

$$=-\overline{\theta\frac{\partial p}{\partial x}}+\nu\left[\frac{\partial^2\overline{\theta u}}{\partial r^2}+[1/r]\frac{\partial\overline{\theta u}}{\partial r}-6\overline{\frac{\partial u}{\partial x}\frac{\partial\theta}{\partial x}}\right] \qquad (4)$$

where the assumptions of isotropy and $\nu=\alpha$ again have been used to simplify the viscous term. The term $6\nu\overline{(\partial u/\partial x)}$

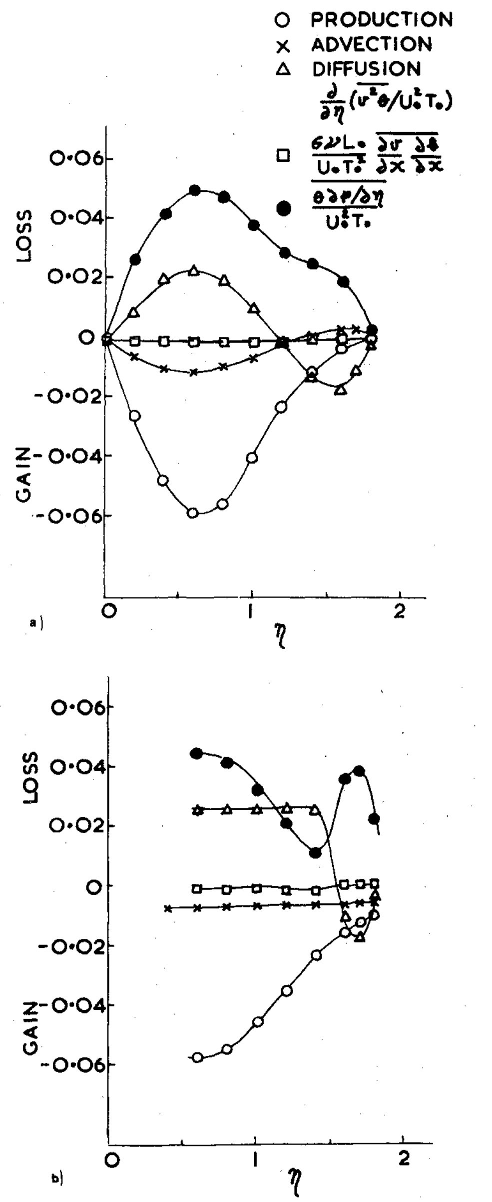

Fig. 3 Budgets: a) radial heat flux $\overline{v\theta}$ and b) $\overline{v\theta_t}$.

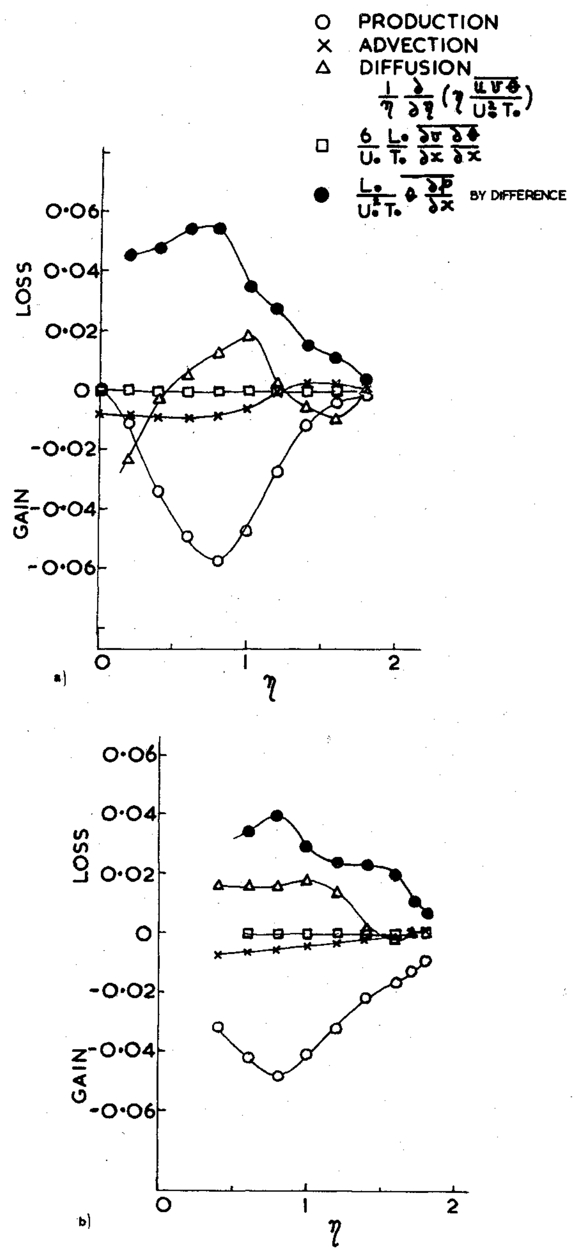

Fig. 4 Budgets: a) axial heat flux $\overline{u\theta}$ and b) $\overline{u\theta_t}$.

$(\overline{\partial\theta/\partial x})$ is required to be zero by isotropy. In the present experiment, this term, although one order magnitude larger than the other two terms within the brackets of Eq. (4), provided negligible contribution to the overall $\overline{u\theta}$ budget (Fig. 4a). The distributions of the various terms in Eq. (4) are similar to the corresponding distributions in Figs. 2a and 3a, except that there appears to be a significant gain of $\overline{u\theta}$ by the diffusion term in the region immediately near the axis. Also, the gain by advection in this region is not negligible. The turbulent budget of Fig. 4b shows the constant loss by diffusion for $\eta<1$, but, in the outer part of the jet, the diffusion is negligible and the shape of the $\overline{\theta\partial p/\partial x}$ term is not too different from its distribution in the conventional $\overline{\theta u}$ budget. The role of the temperature-pressure gradient correlation as essentially the sole mechanism for counteracting the production of $\overline{u\theta}$ or $\overline{v\theta}$ also is demonstrated adequately by the results of Wyngaard et al.[8] in the atmosphere surface layer under a wide range of stability conditions.

V. Discussion

It is useful to consider in this section how the pressure-velocity fluctuations interaction terms inferred by difference from the budgets of $\overline{uv}$, $\overline{\theta u}$, and $\overline{\theta v}$ fit in with some of the models used for these terms in a number of recent and fairly sophisticated calculation methods for turbulent shear flows. It is not our intention here to review these methods in detail, but the reader is referred to the reviews of Reynolds,[10] Bradshaw,[11] and Mellor and Herring.[12] The calculation methods that use rate of transport equations for the Reynolds stresses (or the heat fluxes) have to postulate models for the pressure-velocity correlation and also for the pressure fluctuating strain rate (or pressure-temperature gradient) correlation. Whereas, the former correlation is modeled by a gradient diffusion hypothesis, the latter correlation almost invariably is thought of as a "tendency toward isotropy" term, which, in Cartesian coordinates, is expressed as[13]

$$\overline{p\left[\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right]}\propto(\overline{u_iu_k}-\frac{\delta_{ik}}{3}q^2) \qquad (5)$$

where q^2 is the turbulent energy u_iu_i. By analogy, the pressure-temperature gradient covariance may be taken as proportional to the anisotropy in the flow, viz.

$$\overline{p(\partial\theta/\partial x_k)}\propto\overline{u_k\theta} \qquad (6)$$

One objection to Eq. (5) is that it does not contain any mean velocity gradients, whereas the latter appear exclusively in the Poisson equation for p. The original[14] and more recent[15] attempts by Rotta to model the pressure strain rate interaction do include mean velocity gradients, and, according to Crow's theory[16] for linearized small-scale turbulence, the term that might be added to the right-hand side of Eq. (5) is of the form $c\overline{q^2}(\partial U_i/\partial x_k+\partial U_k/\partial x_i)$, where $c=1/5$. It is not clear what extra term might be added to the right-hand side of Eq. (6), but Deardorff[17] adopts Taulbee's recommendation of a gravitational additive term proportional to $g\overline{\theta^2}/T$ with the proportionality constant equal to $-1/3$.

Donaldson[3] has taken the proportionality constants in Eqs. (5) and (6) as q/Λ_l, where Λ_l (using Donaldson's notation) is taken, in the case of the axisymmetric self-preserving jet with no external stream, proportional to the jet half-velocity radius $r_{0.5}$, say. From Wygnanski and Fiedler's measurements in the axisymmetric jet, $r_{0.5}$ is very roughly proportional to the integral length scale L_u associated with u-component velocity so that, not unexpectedly, $\Lambda_l \propto L_u$. The quickest way to check the validity of Eq. (6) and the magnitude of Λ_l is to start with the $\overline{\theta u}$ budget, where

$$\overline{p\frac{\partial\theta}{\partial x}}=\frac{\partial}{\partial x}(\overline{p\theta})-\overline{\theta\frac{\partial p}{\partial x}}\simeq-\overline{\theta\frac{\partial p}{\partial x}} \tag{7}$$

as $\partial(\overline{p\theta})/\partial x$ is small. Equation (6) then can be rewritten after multiplication by $L_0/U_0^2T_0$:

$$-\left[\overline{\theta\frac{\partial p}{\partial x}}\right]\frac{L_0}{U_0^2T_0}=-\frac{q}{U_0}\frac{L_0}{\Lambda_l}\left[\frac{\overline{u\theta}}{U_0T_0}\right] \tag{8}$$

From the (positive values of $\overline{\theta(\partial p/\partial x)}$ in Fig. 4 and the values of $\overline{u\theta}/U_0T_0$ and $\overline{u^2}/U_0^2$, $\overline{v^2}/U_0^2$ reported in Ref. 1 [q^2 here is assumed equal to $3(\overline{u^2}+\overline{v^2})/2$], the ratio Λ_l/L_0 was calculated and is shown in Fig. 5. The values of Λ_l/L_0 are found to be approximately constant in the inner part ($\eta<1.3$) of the jet, whereas the constancy of Λ_{lt}/L_0, obtained from the turbulent zone averages of the quantities in Eq. (8), extends right across the jet. An average value of $\Lambda_{lt}/r_{0.5}$ is 0.42, which compares favorably with the value of 0.5 chosen by Donaldson for best agreement with the experimental results in the self-preserving jet[6] and the two-dimensional mixing layer.

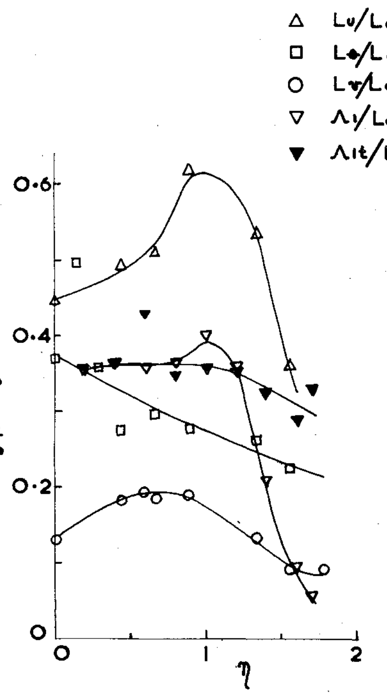

Fig. 5 Distributions of Λ_l and integral length scales L_u, L_v, and L_θ.

Lewellen et al.[18] also used the value of 0.5 in the calculation of the axisymmetric wake results of Chevray.[19] Included in Fig. 5 are the values of the integral length scales for the u, v, and θ fluctuations. These scales were estimated from the low-frequency (1 to 10 Hz) end of the spectra of u, v, θ from the relation $L_x=U\pi[\phi_x]_{\omega=0}/2$, where ϕ_x is the normalized spectral density $\overline{x^2}(\omega)/\overline{x^2}$, ω is the circular frequency $2\pi n$, U is the local velocity, and x stands for either u, v, or θ. The distribution of L_θ is different in shape from that of L_u or L_v. Also, the values of L_θ lie somewhere between those of L_u or L_v and are not too different from those of Λ_l/L_0. It is clear from the distribution of L_u/L_0 that the ratio Λ_l/L_u cannot be assumed constant across the jet. On the jet axis, this ratio is about 0.8 whereas at $\eta=1$ it is approximately 0.65. This range of values is not inconsistent with the value of 0.69, chosen by Donaldson as an average value from the integral length scale distribution in the axisymmetric self-preserving jet.[6]

An assumption common to the calculation procedures of Donaldson[3] and others is that the triple correlations that appear in the velocity diffusion terms are modeled by the gradient diffusion hypothesis. In particular, Donaldson writes

$$\overline{u_iu_j\theta}=-\Lambda_2q[(\partial/\partial x_i)\overline{u_j\theta}+(\partial/\partial x_j)\overline{u_j\theta^2}=-\Lambda_2q(\partial\overline{\theta^2}/\partial x_j)$$

where Λ_2 is a "diffusion"length scale. In the case of the terms $\overline{\theta^2v}$ and $\overline{uv\theta}$ that appear in Eqs. (3) and (4), respectively,the preceding equations yield the approximate relations

$$\frac{\overline{\theta^2v}}{U_0T_0^2}=-\frac{\Lambda_2}{L_0}\frac{q}{U_0}\frac{\partial}{\partial\eta}\left[\frac{\overline{\theta^2}}{T_0^2}\right] \tag{9}$$

$$\frac{\overline{uv\theta}}{U_0^2T_0}=-\frac{\Lambda_2}{L_0}\frac{q}{U_0}\frac{\partial}{\partial\eta}\left[\frac{\overline{u\theta}}{U_0T_0}\right] \tag{10}$$

The values of $\overline{uv\theta}$ (Fig. 6) and $\overline{\theta^2v}$, $\overline{\theta^2}$, and $\overline{u\theta}$ presented in Ref. 1 have been used to obtain estimates of Λ_2 from both Eqs. (9) and (10). The values given in Table 1 do not suggest a constant value of Λ_2, but, the outer part of the jet ($\eta>1$), the values obtained from Eq. (9) are in moderate agreement with those derived from Eq. (10). The values of Λ_{2_t} obtained from the turbulent zone averages of the quantities appearing in Eqs. (9) and (10) show less variation in the region $\eta>1$, especially for Λ_{2_t} derived from Eq. (9), but their magnitude is significantly larger than that of Λ_2. The consistently negative sign for Λ_{2_t} could be avoided by including the sign of $\partial\theta_t^2/\partial r$ in the formulation of Eq. (9). (The values of θ_t^2 presented in Ref. 1 continue to increase in the outer portion of the jet.) The "optimum" value of Λ_2 used by Donaldson was $0.1\Lambda_l$. The results in Table 1 indicate a much larger value, probably in excess of Λ_l if the conditional results are given more weight.

Donaldson also proposes a gradient diffusion model for the pressure diffusion terms. In the present context, this model, in the case of $\overline{pv}$, can be expressed as

$$\eta\frac{\overline{pv}}{U_0^3}=-\frac{q}{U_0}\frac{\Lambda_3}{L_0}\frac{\partial}{\partial\eta}\left[\eta\frac{\overline{v^2}}{U_0^2}\right] \tag{11}$$

where Λ_3 is a diffusion length scale, taken by Donaldson as equal to Λ_2, for best agreement between his calculation and the experimental results. It has been argued, mainly on the basis that little is known about the pressure-velocity covariance, that there is little, if any justification for postulating Eq. (11) separately from the gradient diffusion model of the triple correlation terms given by Eqs. (9) and (10). Mainly as an exercise in providing some estimate of Λ_3, in relation to Λ_2, we used the pressure diffusion values inferred from the turbulent energy budget of Fig. 1b, which we consider to be more reliable than those derived from the conventional q^2 budget, together with the values of $\overline{v_t^2}$ in Ref. 1. As it is the gradient of $(\overline{pv})_t$ which appears in the q_t^2 budget, the accuracy of the

Table 1 Diffusion length scales Λ_2 and Λ_3, microscale λ, and velocity scale q across the jet

η	$\Lambda_2{}^a/L_0$	$\Lambda_{2t}{}^a/L_0$	$\Lambda_2{}^b/L_0$	$\Lambda_{2t}{}^b/L_0$	Λ_{3t}/L_0	λ^c/L_0	$\lambda_t{}^c/L_0$	q/U_0	q_t/U_0
0.2	0.15	0.63	0.44	0.44	...	0.020	0.020	0.524	0.524
0.4	0.10	0.10	0.34	...	0.08	0.020	0.020	0.532	0.532
0.6	0.03	-0.18	0.35	-0.30	-0.07	0.020	0.026	0.524	0.524
0.8	-0.03	-0.51	-0.10	0.64	-0.08	0.021	0.020	0.490	0.498
1.0	1.05	-0.68	0.22	1.34	-0.10	0.024	0.021	0.424	0.458
1.2	0.28	-0.62	0.18	1.71	-0.15	0.025	0.022	0.346	0.416
1.4	0.20	-0.52	0.15	1.18	-0.26	0.022	0.022	0.257	0.373
1.6	0.16	-0.52	0.08	0.65	-0.26	0.016	0.023	0.181	0.323
1.7	0.10	-0.58	...	...	...	0.014	0.025	0.144	0.303

[a] Derived from Eq. (9).
[b] Derived from Eq. (10).
[c] Derived from Eq. (13) with $\Lambda_l/L_0 = 0.35$.

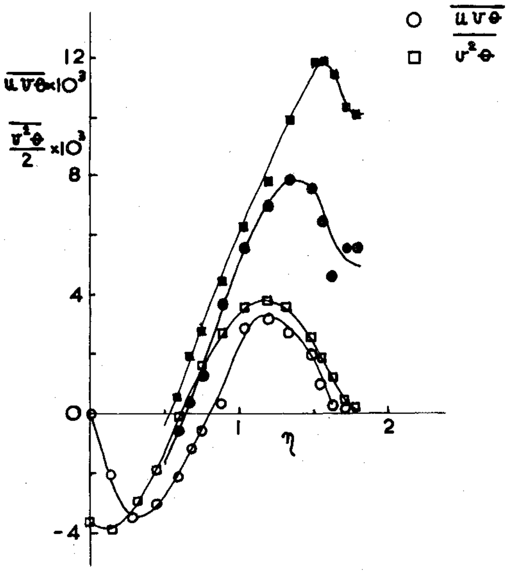

Fig. 6 Triple production correlations $\overline{uv\theta}$ and $\overline{v^2\theta}$. (Closed symbols refer to turbulent zone averages.)

Λ_{3t} values given in Table 1 is poor in view of the double differentiation of $\overline{v_t^2}$ in Eq. (11). If the negative sign of Λ_{3t} is ignored in the tabulation, its absolute magnitude is significantly smaller than that of Λ_{2t} and does not lend much support to the magnitude of Λ_3 chosen by Donaldson. Although the latter author suggests that the calculation is rather insensitive to the choice of Λ_3, even in the case of free shear flows, the results of Fig. 1b do not suggest that the pressure diffusion can be simply neglected.

One further assumption made by Donaldson[3] is that the correlations between velocity and/or temperature gradients that appear in the viscous terms of Eqs. (1-4) can be represented by

$$\overline{(\partial A/\partial x_k)\ (\partial B/\partial x_k)} = \overline{AB}/\lambda^2 \tag{12}$$

where A and B stand for either u, v, w, or θ, and λ is a Taylor-type microscale, taken to be the same for all values of A and B. The value of λ is related to Λ_l by the assumed relation

$$\frac{\lambda}{L_0} = \frac{\Lambda_l/L_0}{[a + b(q/U_0)(\Lambda_l/L_0)(U_0L_0/\nu)]^{1/2}} \tag{13}$$

with $a = 2.5$, $b = 0.125$. Using an average value of $\Lambda_{l_t}/L_0 = 0.35$, the value of λ_t/L_0 (Table 1) is approximately constant ($\simeq 0.02$) across the flow. In the case where $A = B \equiv \theta$, and assuming that the dissipation of temperature fluctuations is isotropic, Eq. (12) yields

$$\lambda^2 = \frac{\overline{\theta^2}}{3\overline{(\partial\theta/\partial x)^2}} \tag{14}$$

where the value of $[\overline{\theta^2}/\overline{(\partial\theta/\partial x)^2}]^{1/2} \equiv \lambda_\theta$, say, sometimes referred to as Corrsin's microscale. The values of λ_θ and $(\lambda_\theta)_t$ in the jet are shown in Fig. 7a. Near the jet axis, the measured value of $\lambda_\theta/L_o = 0.0035$, and the resulting value of λ/L_0 from Eq. (14) is 0.02, in good agreement with the result of Eq. (13). In the outer part of the jet (Fig. 7a), $(\lambda_\theta)_t$ continues to increase as a result of the increase in $\overline{\theta_t^2}$ and the constancy of $\overline{(\partial\theta/\partial x)_t^2}{}^1$. The values of $\lambda_u = [\overline{u^2}/\overline{(\partial u/\partial x)^2}]^{1/2}$ of Fig. 7b are significantly higher than those of λ_θ, and the trend of $(\lambda_u)_t$ in the region $\eta > 1$ is opposite to that of $(\lambda_\theta)_t$. The difference between λ_u and λ_θ is emphasized further in the distributions of turbulence Reynolds and Peclet numbers of Fig. 8. The Peclet number is defined here as $R_{\lambda_\theta} = \overline{u^2}^{1/2}\lambda_\theta/\alpha = R_\lambda Pr(\lambda_\theta/\lambda_u)$, with the Prandtl number $Pr = 0.73$. For the present flow conditions, the Peclet number in the turbulent part of the flow is very nearly constant and equal to 100.

In the case of $A = v$ and $B = \theta$, Eq. (12) yields, again invoking isotropy,

$$3\,\overline{\frac{\partial v}{\partial x}\frac{\partial\theta}{\partial x}} = \frac{\overline{v\theta}}{\lambda^2}$$

$$\frac{6\nu L_0}{T_0U_0^2}\,\overline{\frac{\partial v}{\partial x}\frac{\partial\theta}{\partial x}} = 2\frac{\overline{v\theta}}{U_0T_0}\left[\frac{L_0}{\lambda}\right]^2\frac{\nu}{U_0L_0}$$

The maximum value of $\overline{v\theta}/U_0T_0$ is approximately 0.04, and, for $\lambda/L_0 \simeq 0.02$, the contribution from the preceding term to the $\overline{v\theta}$ budget is about 0.015, which is considerably larger than the measured value in Fig. 3a.

Bradshaw[4] has calculated the development of a thermal boundary layer by converting the exact equation for the rate of change of $\overline{\theta^2}$ along a mean streamline into an equation for $\overline{\theta v}$ by arguments analogous to those used by Bradshaw et al.[20] for converting the equation for $\overline{q^2}$ into one for $\overline{uv}$. The calculation of Bradshaw et al. has been applied, in a slightly modified form, to two-dimensional isothermal turbulent free shear layers.[21] Although it has not, as yet, been extended to axisymmetric flows, it is useful to examine, in the context of the present experimental results, some of the assumptions or empirical functions involved in the conversion of the $\overline{\theta^2}$ equation into an equation for $\overline{\theta v}$. One of the main assumptions relates to the turbulence structure parameter $a_{l\theta}$, defined as

$$a_{l\theta} = \overline{\theta v}/[\overline{\theta^2}^{1/2}(\overline{uv})^{1/2}] \tag{15}$$

relating the thermometric heat flux to the momentum flux. $a_{l\theta}$ is taken simply as constant (or a function of position) across the boundary layer. Figure 9 shows that $a_{l\theta}$ varies between 0.7 and 0.9 for $\eta > 0.4$, and this range of values is not too different from the value of 0.75, obtained by Fulachier[22] in the outer part of the boundary layer. The present values of $(a_{l\theta})_t$ continue to increase towards the outer jet radius as the in-

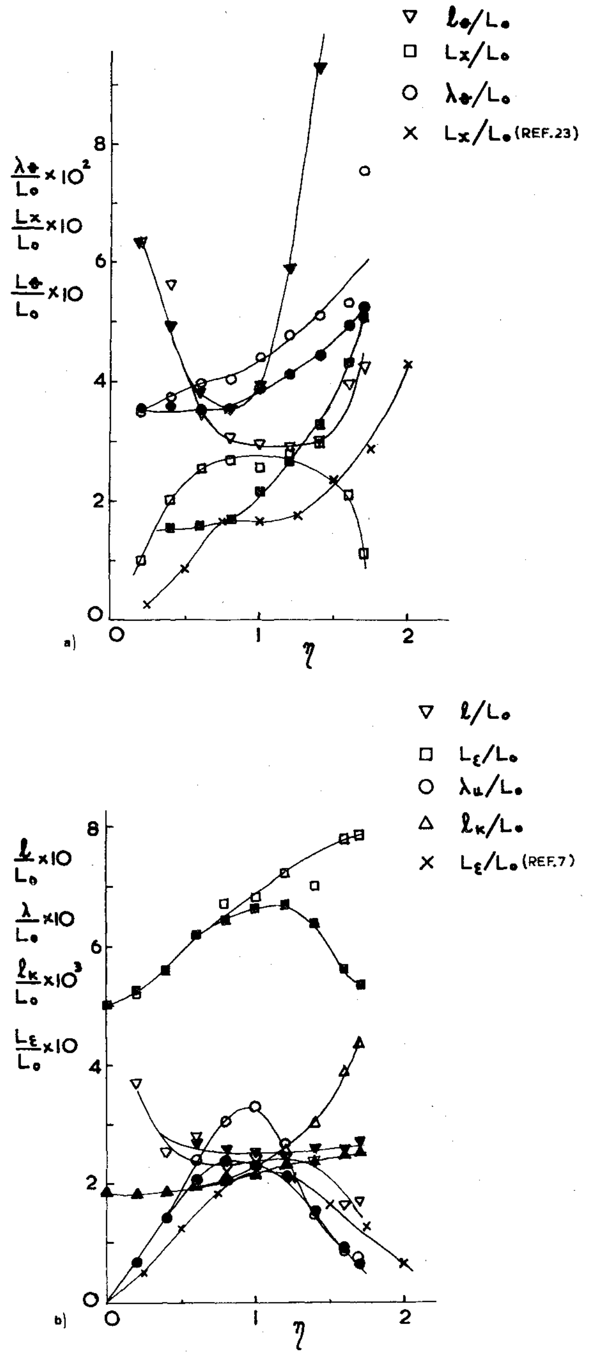

Fig. 7 Length scales: a) temperature fluctuations and b) velocity fluctuations. (Closed symbols refer to turbulent zone averages.)

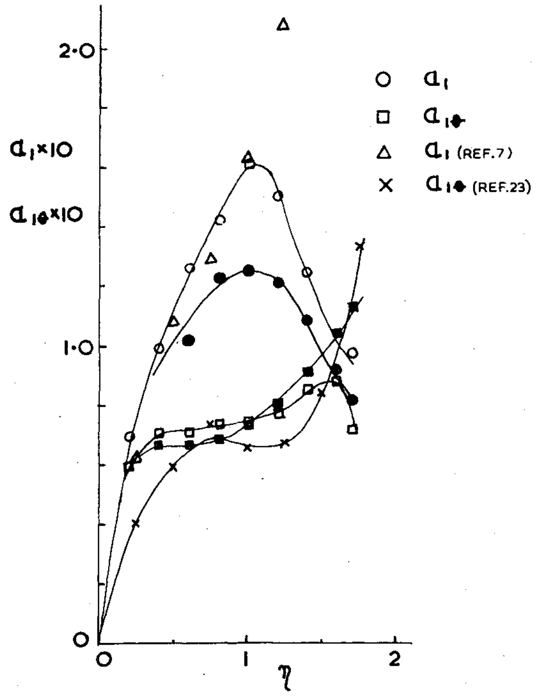

Fig. 9 Structure functions a_l and $a_{l\theta}$. (Closed symbols refer to turbulent zone averages.)

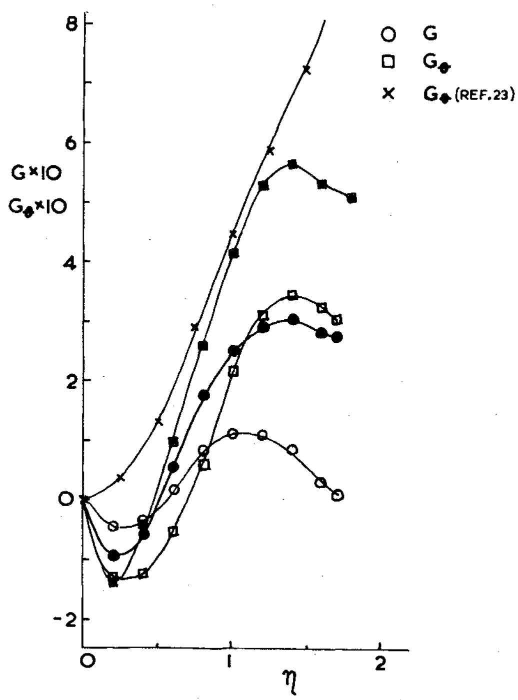

Fig. 10 Diffusion functions G and G_θ. (Closed symbols refer to turbulent zone averages.)

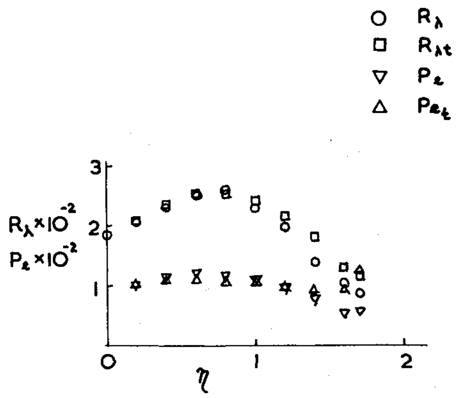

Fig. 8 Turbulent Reynolds and Peclet numbers.

crease in $(\overline{\theta v})_t$ in the region more than offsets the increase in θ_t^2. The results of Freymuth and Uberoi[23] for the wake of a sphere are in reasonable agreement with the present values of $a_{l\theta}$ except at large values of η. The values of $a_l = uv/q^2$ also are shown for comparison in Fig. 9. The values of a_{lt} are consistently lower than those of a_l, and there is only a small region where a_{lt} might be considered constant. Also, the agreement between the present a_l values and those of Ref. 7 is inexplicably poor for $\eta > 1$.

Another assumption is that the diffusion of $\overline{\theta^2}$, or $\partial(\overline{\theta^2 v})/\partial y$, is affected by the large eddies, and $\overline{\theta^2 v}$ is taken as

$$\tfrac{1}{2}\overline{\theta^2 v} = G_\theta \overline{\theta^2}^{1/2} (\overline{uv})^{1/2}_{\max} \qquad (16)$$

where the empirical function G_θ is chosen to fit the experimental data. The present distribution of G_θ differs from that deduced from the results of Freymuth and Uberoi[23] both near the axis of the flow and at the outer edge. The distributions of $G \equiv \frac{1}{2}\overline{q^2 v}/\overline{uv}^{3/2}{}_{\max}$, also shown in Fig. 10 for comparison, qualitatively resemble those of G_θ. It is not our intention here to do a critical comparison between large-scale diffusion and the gradient-type diffusion mentioned in the previous section. Neither of these concepts appears to fit the experimental data over the whose extent of the flow, but the idea of a large-scale transport of $\overline{\theta^2}$ does seem plausible, physically especially in view of the experimental support provided by the conditional measurements of Ref. 1. The insensitivity of the calculation methods to the choice of the diffusion process is exemplified by the moderate success of the Donaldson calculation of a two-dimensional mixing layer, where the importance of the large scale motion is strongly evident.

Perhaps the most critical assumption in Ref. 4 is that the dissipation length scale $L_\chi \equiv (\overline{\theta v})^2/(\overline{uv}^{1/2}\chi)$ is related simply to a length scale of the mean flow, L_0 say. The present values of L_χ/L_0 (Fig. 7a) are significantly larger than those of Freymuth and Uberoi[23] in the region $\eta < 1.2$. This difference also is observed for the values of L_ϵ/L_0 (Fig. 7b), where $L_\epsilon \equiv \overline{uv}^{3/2}/\epsilon$ is a dissipation length scale that may be associated with the integral length scale of the velocity field. The experimental evidence presented in Ref. 24 and the attempts to calculate the development of a jet with a co-flowing stream in Ref. 25 indicate that a simple algebraic relation for L_ϵ is inadequate for this flow and a differential equation for the transport of L_ϵ, and presumably L_χ is required. It also should be mentioned that the difference between L_ϵ and L_χ in the present flow also is reflected in the distributions of the mixing length $\ell = (\overline{uv})^{1/2}/(\partial U/\partial r)$ (Fig. 7b) and the thermal mixing length $\ell_\theta \equiv \overline{\theta v}/(\overline{uv})^{1/2}(\partial T/\partial r)$ (Fig. 7a). Although ℓ_t is fairly constant for $\eta > 0.4$, $\ell_{\theta t}$ has a minimum at $\eta = 0.8$ and increases very rapidly toward the edge of the flow.

VI. Conclusions

The budgets of the conventional Reynolds shear stress and radial and axial heat fluxes all indicate that the production of these quantities is destroyed effectively by the pressure-velocity or pressure-temperature correlations. The budgets for the turbulent zone averages of uv and $v\theta$ indicate an enhanced transport by diffusion of these quantities toward the outer part of the flow. This enhanced diffusion, which is in excess of the production terms in that region of the flow, is balanced again by the pressure-interaction terms. The pressure-temperature gradient correlation that appears in the $\overline{\theta u}$ balance yields a tendency toward isotropy length scale which is in reasonable agreement with the value chosen by Donaldson in his calculation method. The choice of diffusion and dissipation length scales by Donaldson is not, however, in general supported by the data.

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